The Folded Rectangle Construction

Name(s): ________________

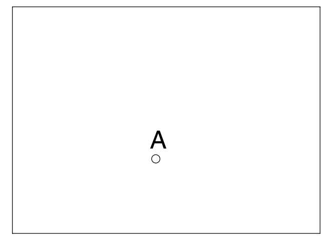
With nothing more than a sheet of paper and a single point on the page, you can create a parabola. No rulers and no measuring required!

Constructing a Physical Model

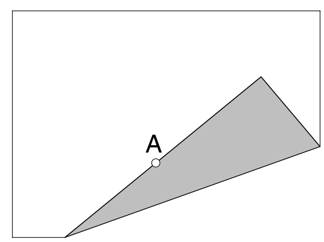
Preparation: *You'll need a rectangular or square piece of wax paper or patty paper. If you don't have these materials, use a plain sheet of paper.*

If you're working in a class, have members place *A* at different distances from the edge. If you're working alone, do this section twice—once with *A* close to the edge, once with *A* farther from the edge.

1. Mark a point *A* approximately one inch from the bottom of the paper and centered between the left and right edges.



2. As shown below right, fold the paper so that a point on the bottom edge lands directly onto point *A*. Make a sharp crease to keep a record of this fold. Unfold the crease.



3. Fold the paper along a new crease so that a different point on the bottom edge lands on point *A*. Unfold the crease and repeat the process.
4. After you've made a dozen or so creases, examine them to see if you spot any emerging patterns.

Mathematicians would describe your set of creases as an **envelope** of creases.

5. Resume creasing the paper. Gradually, you should see a well-outlined curve appear. Be patient—it may take a little while.
6. Discuss what you see with your classmates and compare their folded curves to yours. If you're doing this activity alone, fold a second sheet of paper with point *A* farther from the bottom edge.

Questions

Q1 The creases on your paper seem to form the outline of a parabola. Where do its focus and directrix appear to be?

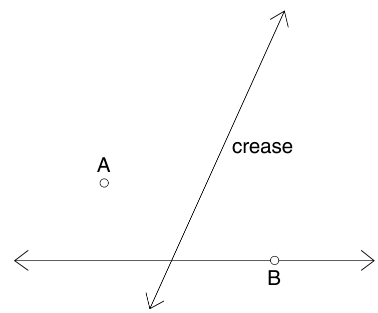
Q2 If you were to move point *A* closer to the bottom edge of the paper and fold another curve, how do you think its shape would compare to the first curve?

Constructing a Sketchpad Model

Fold and unfold. Fold and unfold. Creasing your paper takes some work. Folding one or two sheets is fun, but what would happen if you wanted to continue testing many different locations for point *A*? You'd need to keep starting over with fresh paper, folding new sets of creases.

Sketchpad can streamline your work. With just one set of creases, you can drag point *A* to new locations and watch the crease lines adjust themselves instantaneously.

7. Open a new sketch. Use the **Line** tool to draw a horizontal line near the bottom of the screen. This line represents the bottom edge of the paper.
8. Draw a point *A* above the line, roughly centered between the left and right edges of the screen.
9. Construct a point *B* on the horizontal line.



10. Construct the "crease" formed when point *B* is folded onto point *A*.
11. Drag point *B* along its line. If you constructed your crease line correctly, it should adjust to the new locations of point *B*.
12. Select the crease line and choose **Trace Line** from the Display menu.
13. Drag point *B* along the horizontal line to create a collection of crease lines.
14. Drag point *A* to a different location, then, if necessary, choose **Erase Traces** from the Display menu.
15. Drag point *B* to create another collection of crease lines.

If you don't want your traces to fade, be sure the Fade Traces Over Time box is unchecked on the Color panel of the Preferences dialog box.

Retracing creases for each location of point *A* is certainly faster than folding paper. But we can do better. Ideally, your crease lines should relocate automatically as you drag point *A*. Sketchpad's powerful **Locus** command makes this possible.

16. Turn tracing off for your original crease line by selecting it and once again choosing **Trace Line** from the Display menu.
17. Now select your crease line and point *B*. Choose **Locus** from the Construct menu. An entire set of creases will appear: the locus of crease locations as point *B* moves along its path. If you drag point *A*, you'll see that the crease lines readjust automatically.

Questions

Q3 How does the appearance of the curve change as you move point *A* closer to the horizontal line?

Q4 How does the appearance of the curve change as you move point *A* away from the horizontal line?

Playing Detective

Each crease line on your paper touches the parabola at exactly one point. Another way of saying this is that each crease is *tangent* to the parabola. By engaging in some detective work, you can locate these tangency points and use them to construct just the parabola without its creases.

18. Open the sketch **Folded Rectangle.gsp** in the **Parabola** folder. You'll see a thick crease line and its locus already in place.

19. Drag point *B* and notice that the crease line remains tangent to the parabola. The exact point of tangency lies at the intersection of two lines—the crease line and another line not shown here. Construct this line in your sketch as well as the point of tangency, point *D*.

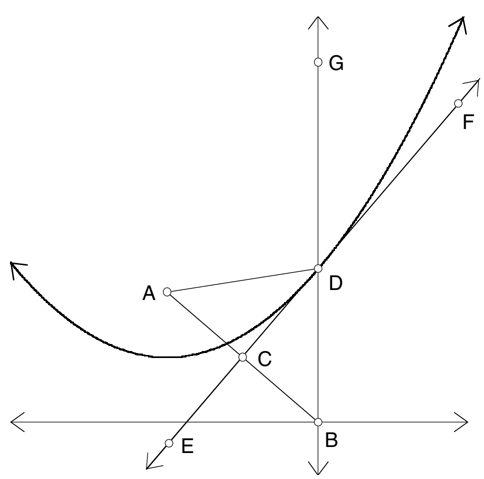
Select the locus and make its width thicker so that it's easier to see.

20. Select point *D* and point *B* and choose **Locus** from the Construct menu. If you've constructed point *D* correctly, you should see a curve appear precisely in the white space bordered by the creases.

How to Prove It

The Folded Rectangle construction seems to generate parabolas. Can you prove that it does? Try developing a proof on your own or work through the following steps and questions.

The picture below should resemble your Sketchpad construction. Line *EF* (the perpendicular bisector of segment *AB*) represents the crease line formed when point *B* is folded onto point *A*. Point *D* sits on the curve itself.



Questions

Q5 Assuming point D traces a parabola, which two segments must you prove equal in length?

Q6 Use a triangle congruence theorem to prove that $\Delta ACD \cong \Delta BCD$.

Remember, a parabola is the set of points equidistant from a fixed point (the focus) and a fixed line (the directrix).

Q7 Use the distance definition of a parabola and the result from Q6 to prove that point D traces a parabola.

Explore More

1. Open the sketch **Tangent Circle.gsp** in the **Parabola** folder. You'll see a circle with center at point C that passes through point A and is tangent to a line at point B. Drag point B. Why does point C trace a parabola?

2. A parabola can be described as an ellipse with one focal point at infinity.

   Open the sketch **Conic Connection.gsp** in the **Parabola** folder. You'll see the ellipse and circle from the Folded Circle construction. Press the *send focal point to "infinity"* button. Point A—a focal point of the ellipse and the center of the circle—will travel far off the screen.

   When the movement stops, examine the result. In what ways does it resemble the Folded Rectangle construction?

3. Use the illustration from your parabola proof to show that $\angle GDF = \angle ADC$. The sketch **Headlights.gsp** in the **Parabola** folder illustrates a nice consequence of this result.

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Prerequisites: For students to complete the proof, they'll need to know the distance definition of a parabola and the SAS triangle congruency theorem.

Sketchpad Proficiency: Intermediate. Students build a perpendicular bisector line and follow the steps of an extended construction. Alternatively, beginner students can view the sketch **Demo Model.gsp** (**Parabola** folder), which contains pre-built models of everything.

Activity Time: 70–80 minutes. If time is short, use the pre-built sketch **Demo Model.gsp** (**Parabola** folder). The activity will likely take less time if you've done the Folded Circle Construction from the Ellipse chapter.

General Notes: The paper-folding technique in this activity is a simple and impressive way to generate parabolas. In fact, since all it requires is a single sheet of paper, this method can be downright addictive. Got a blank sheet of paper? Fold a parabola!

Follow-Up Activities: The Folded Circle Construction (Ellipse chapter) features the same paper-folding technique and could be assigned as an independent project.

Also see the third project listed in Parabola Projects.

Constructing a Physical Model

Q1 Point A is the focus and the bottom edge of the paper is the directrix.

Q2 The curve would appear "narrower."

Constructing a Sketchpad Model

In step 10, students must study the geometry of their crease lines. Specifically, given points B and A, how do you use Sketchpad to construct the "crease" formed when B is folded onto A? (The crease is the perpendicular bisector of segment AB.)

As preparation for this construction step, you might ask students to take a fresh sheet of notebook paper, mark two random points, fold one onto the other, then unfold the paper. What is the geometric relationship of the crease line to the two points?

Q3 As point A approaches the horizontal line, the curve appears "narrower."

Q4 As point A moves away from the horizontal line, the curve appears "wider."

Playing Detective

In step 19, students are asked to construct the point of tangency to their parabola. To do so, construct a line k through point B perpendicular to the horizontal line. The point of tangency lies at the intersection of line k with the crease line.

It's interesting to consider whether a parabola has asymptotes; that is, if there are lines the curve approaches but never crosses. By observing the tangent line as point B moves farther and farther away from the focus, we see that the curve becomes more and more perpendicular to the directrix. So if there are asymptotes, they must be perpendicular to the directrix.

But given any perpendicular, there is a point on it that is equidistant from the focus and the directrix. Thus the curve crosses the line, and the line cannot be an asymptote.

How to Prove It

Q5 You must prove that $AD = DB$.

Q6 Since the crease line is the perpendicular bisector of segment AB, we have $CA = CB$ and $\angle ACD = \angle BCD = 90^\circ$. And, of course, $CD = CD$. Thus, by the SAS triangle congruency theorem, $\Delta ACD \cong \Delta BCD$.

Q7 Since $\Delta ACD \cong \Delta BCD$, corresponding sides AD and DB are equal in length.

Explore More

1. Since the circle is tangent to the line at point B, the radius CB is perpendicular to the line. Thus segment CB represents the distance of point C from the line. Segments CB and CA are both radii of the circle and equal in length. Therefore C traces a parabola with a focus at point A.
2. As point A travels farther and farther off screen, the portion of the circle we see gets straighter and straighter. When point A finally stops moving, we're left with a construction that looks like the Folded Rectangle—a single focus (point B) and a nearly flat directrix.
3. $\angle GDF = \angle BDC$, as they are vertical angles. Since $\Delta ADC \cong \Delta BDC$, we have $\angle ADC = \angle BDC$. Putting these two equalities together gives $\angle GDF = \angle ADC$.